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Transformations of Laguerre 2D polynomials with applications to quasiprobabilities

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Abstract. Laguerre 2D polynomials are defined and their properties are investigated. The Laguerre 2D functions, introduced in [1, 2] are related to the Laguerre 2D polynomials in such a way that they also include the weight function for the orthonormalization of the Laguerre 2D polynomials. A one-parameter group of transformations applicable to certain classes of polynomials and discrete sets of functions is investigated and applied, in particular, to Hermite polynomials and to Laguerre 2D polynomials. These transformations allow us to represent the polynomials of the corresponding classes by superpositions of the same kind of polynomials with stretched arguments. They contain limiting cases with delta functions and their derivatives and lead to regularized representations of the delta functions and their derivatives as demonstrated for Hermite and Laguerre 2D polynomials. Applications of the Laguerre 2D polynomials and 2D functions and their transformations to problems of quantum optics as the representation of quasiprobabilities in the Fock-state basis and by normally and otherwise ordered moments are considered. The inversion of these representations is obtained in all cases. A restricted design of quasiprobabilities should become possible.

1. Introduction

In this paper, we define and investigate Laguerre 2D polynomials which are related to the Laguerre 2D functions, introduced in [1, 2] in such a way that the last include the square root of the weight function which is necessary for the 2D orthonormalization of the first. This orthonormality can be considered as the most important property of the Laguerre 2D functions in applications. All derived relations show that the Laguerre 2D polynomials are in a great analogy to Hermite polynomials and Laguerre 2D functions in analogy to Hermite functions in the 1D case. We introduce the Laguerre 2D polynomials in the next section and discuss some of their most important properties including generating functions. In section 3, we present a method for the derivation of an important kind of transformation formula of polynomials including argument transformations and apply this to Hermite polynomials and Laguerre 2D polynomials. These transformations form a one-parameter group and allow us to represent certain classes of polynomials by superpositions of the same kind of polynomials with stretched arguments and with the stretching factor as a free parameter, however. In section 4, we derive relations which are necessary for the transition to limiting cases in the Laguerre 2D polynomials and which are related to the 2D delta function and its derivatives.

From section 5 onwards, we consider applications of the Laguerre 2D polynomials and 2D functions in quantum optics. The most important application is the representation of

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quasiprobabilities. The first quasiprobability was introduced by Wigner [3] for the purpose of describing quantum mechanics in the most analogous way to classical statistical mechanics by means of distribution functions over the phase space. This method was further developed and later applied to quantum optics in many papers, e.g., [4–10] and is now represented in a large number of monographs and review articles, e.g., [11–20]. The Laguerre 2D polynomials and, alternatively, 2D functions are the most appropriate sets of functions for the representation of the s-ordered class of quasiprobabilities [9, 10] in the Fock-state basis and for their inversion, considered in section 5. Another possibility is to represent the quasiprobabilities by complete sets of ordered moments, in particular, by normally ordered moments [21]. We demonstrate that in this case, the Laguerre 2D polynomials are the most appropriate set of functions for their representation and inversion. The connections between different kinds of ordering of powers of the boson operators [9, 10] can also be expressed by the Laguerre 2D polynomials. Practically, in all 2D problems solved or solvable by usual Laguerre polynomials, one can represent the solutions much better by Laguerre 2D polynomials. The inversion of all these representations is obtained. After our groundwork in sections 3 and 4, we consider and generalize in section 7 transformations which were made in the representation of quasiprobabilities. First, we consider the Peřina–Mišta representation of the Glauber–Sudarshan quasiprobability $P(\alpha, \alpha^*)$ in the Fock-state basis obtained a long time ago [22, 23] and discussed in [24] (see also [12, 13]) and reformulated by using Laguerre 2D functions in [1, 2]. The essential feature of this regularized representation is the substitution of the derivatives of the 2D delta function obtained by a limiting procedure from certain Laguerre polynomials by these Laguerre polynomials themselves without a limiting procedure but therefore with slightly changed effective matrix elements of the density operator. Second, we consider transformations which concern the representation of quasiprobabilities by Laguerre 2D polynomials with stretched arguments. They are related to a representation of the coherent-state (or Husimi–Kano) quasiprobability $Q(\alpha, \alpha^*)$ [4, 8] for Fock states expressed by the Wigner quasiprobabilities $W(\alpha, \alpha^*)$ [3, 6] with stretched arguments for certain incoherent superpositions of the density operators of lower Fock states $\varrho = |m\rangle\langle m|, \ m \leqslant n$ recently proposed in [25] (equation (32)). We derive a transformation which also includes from the beginning, besides the diagonal, the nondiagonal elements of the Fock-state representation of the quasiprobabilities and discuss a different aspect of these transformations as representations by functions for transformed effective matrix elements. By using a more general kind of transformation, as derived in section 3, one can represent arbitrary s-ordered quasiprobabilities by Laguerre 2D polynomials with stretched arguments where the stretching factors are free parameters which can be chosen in appropriate way. These transformations suggest that they can be used for a 'restricted' design of quasiprobabilities in connection with the orthonormality relations for the Laguerre 2D functions as explained in section 7.

2. Introduction of Laguerre 2D polynomials

In this section we introduce Laguerre 2D polynomials which are a very effective means for the representation of many results in quantum optics (quasiprobabilities in Fock-state basis, ordering problems, moments) and, moreover, in other regions of physics. The Laguerre 2D polynomials are very closely related to Laguerre 2D functions introduced in [1] and discussed in [2]. The most relevant property of Laguerre 2D functions is to form a 2D orthonormalized and complete set of functions which are eigenfunctions of the degenerate 2D harmonic oscillator. The Laguerre 2D polynomials are related to the Laguerre 2D functions similarly as the Hermite polynomials to the Hermite functions (eigenfunctions of the 1D harmonic oscillator). The substantial part of the Laguerre 2D polynomials are Laguerre polynomials but their introduction underlines the symmetries as they mostly appear in applications to quantum optics and other regions of physics. The Laguerre 2D polynomials together with products of two Hermite polynomials are special cases of a three-parameter set of polynomials which is equivalent to the two-variable Hermite polynomials [26–31].

We now define the set of Laguerre 2D polynomials $L_{m,n}(z, z^*)$, $(m, n = 0, 1, \ldots)$ as polynomials of two independent variables (z, z^*) which, in applications, are generally a pair of complex conjugated variables in the following way (see also [32]):

$$\begin{aligned} &L_{m,n}(z, z^*) \equiv \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z^m z^{*n} = \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j}, \\ &L_{m,n}(0, 0) = (-1)^n n!\, \delta_{m,n} \qquad \lim_{|z|\to\infty} \frac{L_{m,n}(z, z^*)}{z^m z^{*n}} = 1 \qquad (m, n = 0, 1, \ldots). \end{aligned} \tag{2.1}$$

They are closely related to the Laguerre 2D functions $l_{m,n}(z, z^*)$ according to [1, 2]

$$l_{m,n}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right) \frac{1}{\sqrt{m!n!}} L_{m,n}(z, z^*). \tag{2.2}$$

The Laguerre 2D functions $l_{m,n}(z, z^*)$ can be generated from the 'vacuum' $l_{0,0}(z, z^*)$ in the following way (see the introduction of annihilation and creation operators in [2])

$$\begin{aligned} &l_{m,n}(z, z^*) = \frac{1}{\sqrt{m!n!}} \left(\frac{z}{2} - \frac{\partial}{\partial z^*}\right)^m \left(\frac{z^*}{2} - \frac{\partial}{\partial z}\right)^n l_{0,0}(z, z^*) \\ &l_{0,0}(z, z^*) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{zz^*}{2}\right). \end{aligned} \tag{2.3}$$

By splitting from $l_{0,0}(z, z^*)$ a factor $\exp(zz^*/2)$ and by using the commutation rules of $\partial/\partial z^*$ and $\partial/\partial z$ with $\exp(zz^*/2)$, one immediately obtains the following alternative representations of Laguerre 2D polynomials:

$$\begin{aligned} L_{m,n}(z, z^*) &= (-1)^{m+n} \exp(zz^*) \frac{\partial^{m+n}}{\partial z^{*m} \partial z^n} \exp(-zz^*) \\ &= \left(z - \frac{\partial}{\partial z^*}\right)^m \left(z^* - \frac{\partial}{\partial z}\right)^n 1. \end{aligned} \tag{2.4}$$

The representation of the Laguerre 2D polynomials by the usual Laguerre polynomials $L_n^\alpha(u)$ in their modern definition [26, 33–35] is given by

$$L_{m,n}(z, z^*) = (-1)^n n! z^{m-n} L_n^{m-n}(zz^*) = (-1)^m m! z^{*n-m} L_m^{n-m}(zz^*). \tag{2.5}$$

The relation of the Laguerre 2D polynomials to the Laguerre 2D functions is that the latter take into account the (square root of the) weight function for the orthonormalization of the Laguerre 2D polynomials.

We now discuss some important properties of the Laguerre 2D polynomials which can be derived from the properties of the usual Laguerre polynomials or from the properties of Laguerre 2D functions considered in [2] or directly from their definition by (2.1) or (2.4). The symmetry of the Laguerre 2D polynomials is

$$L_{m,n}(-z, -z^*) = (-1)^{m+n} L_{m,n}(z, z^*) \qquad L_{m,n}(z, z^*) = \left(L_{n,m}(z, z^*)\right)^* = L_{n,m}(z^*, z) \tag{2.6}$$

showing that the Laguerre 2D polynomials $L_{m,n}(z, z^*)$ possess the parity $(-1)^{m+n}$. The lowering of the indices is obtained by the following operations of differentiation

$$\frac{\partial}{\partial z} L_{m,n}(z, z^*) = m L_{m-1,n}(z, z^*) \qquad \frac{\partial}{\partial z^*} L_{m,n}(z, z^*) = n L_{m,n-1}(z, z^*) \tag{2.7}$$

and the raising of the indices by the operations

$$
\begin{aligned}
&-\exp(zz^*)\frac{\partial}{\partial z^*}\exp(-zz^*)L_{m,n}(z,z^*) = \left(z - \frac{\partial}{\partial z^*}\right)L_{m,n}(z,z^*) = L_{m+1,n}(z,z^*)\\
&-\exp(zz^*)\frac{\partial}{\partial z}\exp(-zz^*)L_{m,n}(z,z^*) = \left(z^* - \frac{\partial}{\partial z}\right)L_{m,n}(z,z^*) = L_{m,n+1}(z,z^*).
\end{aligned}
\tag{2.8}
$$

The Laguerre 2D polynomials satisfy the following recursion relations derived from (2.8) by using (2.7)

$$
\begin{aligned}
L_{m+1,n}(z,z^*) &= zL_{m,n}(z,z^*) - nL_{m,n-1}(z,z^*)\\
L_{m,n+1}(z,z^*) &= z^*L_{m,n}(z,z^*) - mL_{m-1,n}(z,z^*).
\end{aligned}
\tag{2.9}
$$

One of the most simple generating functions of the Laguerre 2D polynomials together with its derivation by using (2.1) is [32]

$$
\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}L_{m,n}(z,z^*) &= \exp\left(-\frac{\partial^2}{\partial z\partial z^*}\right)\sum_{m=0}^{\infty}\frac{(sz)^m}{m!}\sum_{n=0}^{\infty}\frac{(tz^*)^n}{n!}\\
&= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{\partial^{2j}}{\partial z^j\partial z^{*j}}\exp(sz+tz^*)\\
&= \exp(sz+tz^*-st).
\end{aligned}
\tag{2.10}
$$

It is analogous to the most simple generating function for the Hermite polynomials. For products of two Hermite polynomials, this generating function takes on the form

$$
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}H_m(x)H_n(y) = \exp(2(sx+ty)-(s^2+t^2)).
\tag{2.11}
$$

In both cases (2.10) and (2.11), we have a special symmetric quadratic form of the two variables (s, t) in the exponent together with a linear form. By using the 2D unity matrix and the two symmetric matrices from the three Pauli spin matrices (σ_x and σ_z are symmetric whereas σ_y is antisymmetric), the general class of symmetric quadratic forms of two variables can be mapped onto a complex three-component vector and can be parametrized by this vector (see [36] for a similar case). This is one possible key for the unification of products of two Hermite polynomials and of Laguerre 2D polynomials and to establish the connection to two-variable Hermite polynomials.

Similar to (2.10), but more difficult, is the derivation of the following generating function for squared products of Laguerre 2D polynomials:

$$
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}L_{m,n}(z,z^*)L_{n,m}(w,w^*) = \frac{1}{1-st}\exp\left(\frac{szw^*+twz^*-st(zz^*+ww^*)}{1-st}\right).
\tag{2.12}
$$

It is analogous to the formula of Mehler for Hermite polynomials ([26] equation (10.13.22); derivation in [33])

$$
\sum_{n=0}^{\infty}\frac{t^n}{2^n n!}H_n(x)H_n(y) = \frac{1}{\sqrt{1-t^2}}\exp\left(\frac{2txy-t^2(x^2+y^2)}{1-t^2}\right).
\tag{2.13}
$$

The sum in (2.12) converges in the usual sense only for $st < 1$ but, surely, this equality can be extended to more general cases if one considers both sides as generalized functions or linear continuous functionals. That means the left-hand side in the sense of weak convergence to the right-hand side. By transition to the Laguerre 2D functions in (2.12) according to (2.2) and by substituting $s = t = 1 - \varepsilon/2$, one derives the following identity in the limiting case $\varepsilon \to +0$:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} l_{m,n}(z, z^*) l_{n,m}(w, w^*) = \exp\left(\frac{(z-w)(z^*-w^*)}{2}\right) \lim_{\varepsilon\to+0} \frac{1}{\pi\varepsilon} \times \exp\left(-\frac{(z-w)(z^*-w^*)}{\varepsilon}\right) = \delta(z-w, z^*-w^*) \tag{2.14}$$

where $\delta(z, z^*)$ denotes the 2D delta function in (z, z^*)-variables ($\delta(z, z^*) \equiv \delta(x)\delta(y)$). This is the completeness relation for the Laguerre 2D functions (see equation (3.10) in [2])). The analogous completeness relation for the Hermite polynomials can be derived from (2.13) by a limiting procedure. By substitutions of the variables together with special cases of vanishing of variables or limiting procedures in (2.12), one can derive more identities of the form of generating functions (see [26], equations (10.13.19–21) for Hermite polynomials).

A very interesting category of relations for the Laguerre 2D polynomials is the representation of these polynomials by superpositions of the same set of polynomials, however, with stretched arguments. We derive these transformations from a unique point of view and with similar transformations for Hermite and other polynomials in the next section.

3. Transformation formulae for some classes of polynomials and functions

In this section we derive transformation formulae which can be applied to some important classes of polynomials, in particular to the Hermite polynomials and to the Laguerre 2D polynomials. A related but different kind of transformation can be applied to countably infinite sets of functions. Let us begin with the derivation of an auxiliary formula, the meaning of which becomes immediately clear after considering examples.

Suppose that we have a set of functions of the real or complex variable z (usually polynomials) which we denote by $f_n(0; z)$, $(n = 0, 1, 2, \ldots)$. Suppose, furthermore, that $f_n(0; z)$ vanish automatically by setting $n = -1, -2, \ldots$ or by definition. Now, we transform this set of functions or polynomials with a real continuous parameter ε into a new set $f_n(\varepsilon; z)$ in the following way:

$$f_n(\varepsilon; z) \equiv \sum_{k=0}^{[n/\mu]} \frac{(-\varepsilon)^k}{k!} f_{n-\mu k}(0; z) \qquad \mu = 1, 2, \ldots. \tag{3.1}$$

The choice of the notation ε for the parameter should suggest that we can use it as a small parameter in limiting procedures and indeed for $\varepsilon = 0$, one obtains the identical transformation but it is not necessarily a small parameter. It merely determines a one-parameter group of transformations with ε as the additive parameter. The notation $[n/\mu]$ means the integer part of n/μ. The integer μ is fixed in applications in the desired way. The inversion of the transformation (3.1) can be readily obtained and possesses the form

$$f_n(0; z) = \sum_{k=0}^{[n/\mu]} \frac{\varepsilon^k}{k!} f_{n-\mu k}(\varepsilon; z) \tag{3.2}$$

that can be considered as a consequence of the group property and the additivity of the parameter ε. The proof can be given by inserting (3.2) into (3.1) and by reordering the double sum with evaluation of the arising inner sum as follows:

$$f_n(\varepsilon; z) = \sum_{k=0}^{[n/\mu]} \sum_{l=0}^{[n/\mu-k]} \frac{(-\varepsilon)^k \varepsilon^l}{k!l!} f_{n-\mu(k+l)}(\varepsilon; z) = \sum_{j=0}^{[n/\mu]} \frac{\varepsilon^j}{j!} f_{n-\mu j}(\varepsilon; z) \sum_{k=0}^{j} \frac{(-1)^k j!}{k!(j-k)!}$$

$$= \sum_{j=0}^{[n/\mu]} \frac{\varepsilon^j}{j!} f_{n-\mu j}(\varepsilon; z)\delta_{j,0} = f_n(\varepsilon; z). \tag{3.3}$$

Thus, we have proved that (3.1) and (3.2) form a pair of a transformations together with its inverse transformation. By the substitution $f_n(0; z) \equiv g_n(0; z)/n!$, the transformation formulae (3.1) and (3.2) take on the form

$$g_n(\varepsilon; z) = \sum_{k=0}^{[n/\mu]} \frac{(-\varepsilon)^k n!}{k!(n-\mu k)!} g_{n-\mu k}(0; z) \qquad g_n(0; z) = \sum_{k=0}^{[n/\mu]} \frac{\varepsilon^k n!}{k!(n-\mu k)!} g_{n-\mu k}(\varepsilon; z). \tag{3.4}$$

For $\mu = 1$ and $g_n(z) = g_n$ independent on z, it was last refered to as a Bernoulli transformation [37]. The coefficients in this transformation are the binomial coefficients and the binomial distribution is also called the Bernoulli distribution. However, contrary to the well known notions of a Bernoulli distribution and of Bernoulli trials, I could not find the notion of a Bernoulli transformation in the textbook literature about probability theory and it seems that this notion was introduced (or rediscovered?) only recently.

We now consider another but similar kind of transformation as in (3.1) with weighted summation of a set of functions $f_n(z; 0)$ over indices above an arbitrary index n of the following kind and written in the form most near to our later application:

$$f_n(\varepsilon; z) \equiv \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} f_{n+\nu k}(0; z) \qquad \nu = 1, 2, \ldots. \tag{3.5}$$

The inversion of this transformation possesses the form

$$f_n(0; z) = \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k!} f_{n+\nu k}(\varepsilon; z). \tag{3.6}$$

The proof is in full analogy to (3.3) and we do not write it down. In the next section, we consider applications of this transformation.

As the most important example of the application of (3.1) and (3.2), we consider the polynomials $f_n(0; z) = z^n/n!$. For $\mu = 1$ this leads to the binomial formula with $f_n(\varepsilon; z) = (z-\varepsilon)^n/n!$ and its inversion. The case $\mu = 2$ with the same polynomials leads to the Hermite polynomials $H_n(z)$ which alternatively can be defined by [36, 38, 39] (basic definition, e.g., [26, 33, 34])

$$H_n(z) \equiv \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial z^2}\right)(2z)^n = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!}(2z)^{n-2k}. \tag{3.7}$$

By applying (3.1) with $f_n(0; z) = (2z)^n/n!$ and $\mu = 2$, one finds the relation

$$\frac{(\sqrt{\varepsilon})^n}{n!} H_n\left(\frac{z}{\sqrt{\varepsilon}}\right) = \sum_{k=0}^{[n/2]} \frac{(-\varepsilon)^k}{k!} \frac{(2z)^{n-2k}}{(n-2k)!}. \tag{3.8}$$

Now, one can immediately write down the inversion according to (3.2) which provides

$$\frac{(2z)^n}{n!} = \sum_{k=0}^{[n/2]} \frac{\varepsilon^k}{k!} \frac{(\sqrt{\varepsilon})^{n-2k}}{(n-2k)!} H_{n-2k}\left(\frac{z}{\sqrt{\varepsilon}}\right) \tag{3.9}$$

and which can be written in the final form

$$z^n = \left(\frac{\sqrt{\varepsilon}}{2}\right)^n \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} H_{n-2k}\left(\frac{z}{\sqrt{\varepsilon}}\right). \tag{3.10}$$

Thus we have derived a formula for powers z^n expressed by superpositions of Hermite polynomials with any desired stretch factor in the argument. In particular, by choosing $\varepsilon = 1$, one obtains

$$z^n = \frac{1}{2^n} \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} H_{n-2k}(z) = \frac{1}{2^n} \exp\left(\frac{1}{4}\frac{\partial^2}{\partial z^2}\right) H_n(z). \tag{3.11}$$

It is clear that by possessing this formula, one can easily find the form (3.10) by argument scaling but first one has to derive one such a formula. The cases $\mu = 3, 4, \ldots$ and $f_n(0; z) = z^n/n!$ lead to arbitrary-order Hermite polynomials [39, 40] which have not been introduced up to now as standard polynomials in mathematical tables and which have been rarely applied until now. By setting $z = x + \mathrm{i}y$ on the left-hand side in (3.11), one can first apply the binomial formula and then (3.11) separately to powers of x and y that leads to

$$(x + \mathrm{i}y)^n = \frac{1}{2^n} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \mathrm{i}^k H_{n-k}(x) H_k(y). \tag{3.12}$$

The application of the definition of the Hermite polynomials in connection with (3.9) allows us, in a simple way, to derive the connection of Hermite polynomials to Hermite polynomials with stretched argument as follows:

$$H_n(z) = (\sqrt{\varepsilon})^n \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!} \left(\frac{\varepsilon - 1}{\varepsilon}\right)^k H_{n-2k}\left(\frac{z}{\sqrt{\varepsilon}}\right). \tag{3.13}$$

The identical representation corresponds here to $\varepsilon = 1$. It is clear that the most of the derived formulae for Hermite polynomials are not totally new and that they can be specialized from known formulae and, most importantly, the general point of view of their derivation is original and can be further generalized. If we choose $f_n(0; z) = z^n/\sqrt{n!}$ and $\mu = 1, 2, \ldots,$ we arrive at some very unorthodox polynomials. If we choose $f_n(0; z) = z^n/n!^2$ and $\mu = 1$ in (3.1), we come to relations connected with Laguerre polynomials which can, however, be derived in a more general form from the 2D generalization of (3.1) and (3.2) that we next consider.

We now consider polynomials $f_{m,n}(0; z, z^*)$ of two variables (z, z^*) which in application to quantum optics are mostly a pair of complex conjugated variables. However, they can be other independent and also real variables and are in such cases better denoted by, e.g., (x, y). The corresponding transformation to (3.1) is now

$$f_{m,n}(\varepsilon; z, z^*) \equiv \sum_{j=0}^{[\{m,n\}/\mu]} \frac{(-\varepsilon)^j}{j!} f_{m-\mu j, n-\mu j}(0; z, z^*) \qquad \mu = 1, 2, \ldots \tag{3.14}$$

together with its inversion

$$f_{m,n}(0; z, z^*) = \sum_{j=0}^{[\{m,n\}/\mu]} \frac{\varepsilon^j}{j!} f_{m-\mu j, n-\mu j}(\varepsilon; z, z^*). \tag{3.15}$$

The proof is fully analogous to (3.3) and we do not write it down. The corresponding transformation with summation over the upper indices for any index pair (m, n) is given by

$$f_{m,n}(\varepsilon; z, z^*) \equiv \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} f_{m+\nu k, n+\nu k}(0; z, z^*) \qquad \nu = 1, 2, \ldots \tag{3.16}$$

with the inversion

$$f_{m,n}(0; z, z^*) = \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{k!} f_{m+\nu k, n+\nu k}(\varepsilon; z, z^*). \tag{3.17}$$

The proof is again analogous to the proof for the transformation (3.14) and (3.15) or (3.1) and (3.2) in the 1D case. We consider applications of this transformation in the next section.

A very important example of the application of (3.14) and (3.15) is $f_{m,n}(0; z, z^*) = (z^m/m!)(z^{*n}/n!)$ with $\mu = 1$ leading to Laguerre 2D polynomials and we recall their basic definition (2.1)

$$L_{m,n}(z, z^*) \equiv \exp\left(-\frac{\partial^2}{\partial z \partial z^*}\right) z^m z^{*n} = \sum_{j=0}^{\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z^{m-j} z^{*n-j} \tag{3.18}$$

which is in great analogy to definition (3.7) of the Hermite polynomials. If we now choose in (3.14) $f_{m,n}(0; z, z^*) = (z^m/m!)(z^{*n}/n!)$ and $\mu = 1$, we can write it by using the Laguerre 2D polynomials in the following specialized form:

$$\frac{(\sqrt{\varepsilon})^{m+n}}{m!n!} L_{m,n}\left(\frac{z}{\sqrt{\varepsilon}}, \frac{z^*}{\sqrt{\varepsilon}}\right) = \sum_{j=0}^{\{m,n\}} \frac{(-\varepsilon)^j}{j!} \frac{z^{m-j} z^{*n-j}}{(m-j)!(n-j)!} \tag{3.19}$$

and its inversion can immediately be taken from (3.15) providing

$$\frac{z^m z^{*n}}{m!n!} = \sum_{j=0}^{\{m,n\}} \frac{\varepsilon^j}{j!} \frac{(\sqrt{\varepsilon})^{m+n-2j}}{(m-j)!(n-j)!} L_{m-j,n-j}\left(\frac{z}{\sqrt{\varepsilon}}, \frac{z^*}{\sqrt{\varepsilon}}\right). \tag{3.20}$$

This can be written in the following final form:

$$z^m z^{*n} = (\sqrt{\varepsilon})^{m+n} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} L_{m-j,n-j}\left(\frac{z}{\sqrt{\varepsilon}}, \frac{z^*}{\sqrt{\varepsilon}}\right). \tag{3.21}$$

In particular, for $\varepsilon = 1$ one obtains

$$z^m z^{*n} = \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} L_{m-j,n-j}(z, z^*) \tag{3.22}$$

which, in compact form, is the convolution (see (2.1) and (2.7); notation $*$ for convolution)

$$z^m z^{*n} = \exp\left(\frac{\partial^2}{\partial z \partial z^*}\right) L_{m,n}(z, z^*) = \frac{1}{\pi} \exp(-zz^*) * L_{m,n}(z, z^*). \tag{3.23}$$

Similar to the case of Hermite polynomials in (3.13), one can derive the following more general formula which represents Laguerre 2D polynomials by the superposition of Laguerre 2D polynomials with stretched argument

$$L_{m,n}(z, z^*) = (\sqrt{\varepsilon})^{m+n} \sum_{j=0}^{\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} \left(\frac{\varepsilon - 1}{\varepsilon}\right)^j L_{m-j,n-j}\left(\frac{z}{\sqrt{\varepsilon}}, \frac{z^*}{\sqrt{\varepsilon}}\right). \tag{3.24}$$

It gives the identical representation for $\varepsilon = 1$.

We mention here that the Laguerre 2D polynomials in real representation are connected with the Hermite polynomials in the following way (derivation in [2], equations (6.11) and (6.12) or (6.5) and (6.6)):

$$L_{m,n}(x + \mathrm{i}y, x - \mathrm{i}y) = \frac{1}{2^{m+n}} \sum_{j=0}^{m+n} (\mathrm{i}2)^j P_j^{(m-j,n-j)}(0) H_{m+n-j}(x) H_j(y) \tag{3.25}$$

with the inversion

$$H_m(x) H_n(y) = (-\mathrm{i})^n \sum_{j=0}^{m+n} 2^j P_j^{(m-j,n-j)}(0) L_{m+n-j,j}(x + \mathrm{i}y, x - \mathrm{i}y) \tag{3.26}$$

where $P_j^{(\alpha,\beta)}(u)$ denotes the Jacobi polynomials taken here with the argument $u = 0$.

4. Limiting procedures with Laguerre 2D polynomials and Hermite polynomials

We derive in this section some relations for the purpose of making limiting procedures in formulae with Laguerre 2D functions or 2D polynomials. Our starting point is the following moment series expansion of a Gaussian function derived, e.g., in [1] (equation (B.14)), see also [24]

$$\frac{1}{\pi\varepsilon}\exp\left(-\frac{zz^*}{\varepsilon}\right) = \exp\left(\varepsilon\frac{\partial^2}{\partial z\partial z^*}\right)\delta(z,z^*) = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\frac{\partial^{2k}}{\partial z^k\partial z^{*k}}\delta(z,z^*). \quad (4.1)$$

We call such expansions moment series expansions because the coefficients in front of the derivatives of the delta functions are determined by the moments of the function. They can be proved by Fourier transformation. By using the equivalent definition of the Laguerre 2D polynomials in (2.4), one readily obtains the following generalization:

$$\frac{1}{\pi\varepsilon}\exp\left(-\frac{zz^*}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^{m+n} L_{m,n}\left(\frac{z}{\sqrt{\varepsilon}},\frac{z^*}{\sqrt{\varepsilon}}\right) = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}\frac{\partial^{m+n+2k}}{\partial z^{*m+k}\partial z^{n+k}}\delta(z,z^*). \quad (4.2)$$

This relation is of the form of the transformation (3.16) and we can immediately write down its inversion according to (3.17)

$$\frac{\partial^{m+n}}{\partial z^{*m}\partial z^n}\delta(z,z^*) = \frac{1}{\pi\varepsilon}\exp\left(-\frac{zz^*}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^{m+n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}L_{m+k,n+k}\left(\frac{z}{\sqrt{\varepsilon}},\frac{z^*}{\sqrt{\varepsilon}}\right). \quad (4.3)$$

Thus, we have obtained a representation of the derivatives of the 2D delta function $\delta(z,z^*)$ by an infinite series over regular functions. I have checked by graphical representation, with a computer, the degree of approximation of the 2D delta function in this relation for various positive values of the parameter ε and various numbers of truncation of the infinite sum (practically, one has to check this only for $\delta(z,z^*)$ because the correctness of the derivatives can then be directly checked by their calculation from the expression for $\delta(z,z^*)$ within (4.3) by using (2.4)). The relations (4.1)–(4.3) are exact identities in the sense of (weak) convergence of generalized functions (see, e.g., [42–44] for notion of convergence of generalized functions).

By making limiting procedures $\varepsilon\to 0$ in the above relations, only the sum terms to $k=0$ survive on the right-hand sides. This leads to

$$\frac{\partial^{m+n}}{\partial z^{*m}\partial z^n}\delta(z,z^*) = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\exp\left(-\frac{zz^*}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^{m+n}L_{m,n}\left(\frac{z}{\sqrt{\varepsilon}},\frac{z^*}{\sqrt{\varepsilon}}\right) \quad (4.4)$$

and is one possibility of the infinite manifold of possibilities to represent the derivatives of a 2D delta function by a limiting procedure from regular functions. The derived relations are important in applications, for example, to quasiprobabilities for the transition to limiting cases. Since the corresponding formulae for Hermite polynomials are rarely available in the literature we will give them here. In analogy to (4.2), one obtains

$$\frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^n H_n\left(\frac{x}{\sqrt{\varepsilon}}\right) = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!2^{2k}}\delta^{(n+2k)}(x). \quad (4.5)$$

One can look to this relation as a transformation of $\delta^{(n)}(x)/2^n$ which satisfies the form of the more general transformation (3.6) with $\nu=2$. From its inversion in (3.7), one finds the following inversion of (4.5):

$$\delta^{(n)}(x) = \frac{1}{\sqrt{\pi\varepsilon}}\exp\left(-\frac{x^2}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,2^{2k}}H_{n+2k}\left(\frac{x}{\sqrt{\varepsilon}}\right). \quad (4.6)$$

This is a regularized representation of the derivatives of the delta function by an infinite series with a (positive) parameter ε. I have intensively checked this relation for $\delta(x)$ by a computer and did not find any contradictions (the derivatives $\delta^{(n)}(x)$ within (4.6) are consistent with $\delta(x)$).

From the relations (4.5) or (4.6), one finds the following representation of the derivatives of the delta function by a limiting procedure:

$$\delta^{(n)}(x) = \lim_{\varepsilon\to 0} \frac{1}{\sqrt{\pi\varepsilon}} \exp\left(-\frac{x^2}{\varepsilon}\right)\left(-\frac{1}{\sqrt{\varepsilon}}\right)^n H_n\left(\frac{x}{\sqrt{\varepsilon}}\right). \tag{4.7}$$

Thus we have obtained the most important relations for making limiting procedures in relations written by means of the Laguerre 2D polynomials and Hermite polynomials that is so often desirable in applications.

Another kind of relation with Laguerre 2D polynomials which often appears in applications is the 'disentanglement' of products of Gaussian factors with delta functions and their derivatives. One has the following relation in the 2D case:

$$\begin{aligned}\exp(-\mu zz^*)\frac{\partial^{m+n}}{\partial z^m\partial z^{*n}}\delta(z,z^*) &= \sum_{j=0}^{\{m,n\}} \frac{(-\mu)^j m!n!}{j!(m-j)!(n-j)!}\frac{\partial^{m+n-2j}}{\partial z^{m-j}\partial z^{*\,n-j}}\delta(z,z^*)\\ &= (\sqrt{\mu})^{m+n}L_{m,n}\left(\frac{1}{\sqrt{\mu}}\frac{\partial}{\partial z},\frac{1}{\sqrt{\mu}}\frac{\partial}{\partial z^*}\right)\delta(z,z^*).\end{aligned} \tag{4.8}$$

It was derived in [1] together with the more general relation for the 'disentanglement' of products of functions with derivatives of delta functions which possesses the form (see also [41], equation (8.20))

$$f(z,z^*)\frac{\partial^{m+n}}{\partial z^m\partial z^{*n}}\delta(z,z^*) = \sum_{k=0}^{m}\sum_{l=0}^{n}\frac{(-1)^{k+l}m!n!}{k!(m-k)!l!(n-l)!}\frac{\partial^{k+l}f}{\partial z^k\partial z^{*l}}(0,0)\frac{\partial^{m+n-k-l}}{\partial z^{m-k}\partial z^{*n-l}}\delta(z,z^*). \tag{4.9}$$

The corresponding relation to (4.8) in the 1D case is

$$\exp(-\mu x^2)\delta^{(n)}(x) = \sum_{k=0}^{[n/2]}\frac{(-\mu)^k n!}{k!(n-2k)!}\delta^{(n-2k)}(x) = (\sqrt{\mu})^n H_n\left(\frac{1}{2\sqrt{\mu}}\frac{\partial}{\partial x}\right)\delta(x). \tag{4.10}$$

We have applied (4.8) and (4.10) in [1] to obtain a 'disentangled' form of the Sudarshan representation [7] of the Glauber–Sudarshan quasiprobability (see also [19,20] for Fock states).

5. Quasiprobabilities in Fock-state representation and their inversion

We begin in this section with the consideration of applications of Laguerre 2D polynomials and 2D functions in quantum optics. First, we consider the Fock-state representation of the class of quasiprobabilites $F_{\boldsymbol{r}}(\alpha,\alpha^*)$ with the vector parameter $\boldsymbol{r} \equiv (r_1, r_2, r_3)$ specialized as $\boldsymbol{r} = (0,0,r)$ (s-parametrized class of quasiprobabilities [9, 10], $r_3 = r = -s$) by means of the Laguerre 2D polynomials.

The mentioned class of quasiprobabilities $F_{(0,0,r)}(\alpha,\alpha^*)$ takes on the following form in Fock-state representation (e.g., [1]):

$$\begin{aligned}F_{(0,0,r)}(\alpha,\alpha^*) &= \frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\\ &\quad\times\frac{1}{\sqrt{m!n!}}\left(\sqrt{\frac{1-r}{1+r}}\right)^{m+n} L_{n,m}\left(\frac{2\alpha}{\sqrt{1-r^2}},\frac{2\alpha^*}{\sqrt{1-r^2}}\right).\end{aligned} \tag{5.1}$$

The inversion of this relation that means the determination of the matrix elements $\langle m|\varrho|n\rangle$ of the density operator ϱ from the quasiprobabilities can be made in the most simple way by using the orthonormality and completeness of the Laguerre 2D functions with the following result written by Laguerre 2D polynomials ($(\mathrm{i}/2)\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^* = \mathrm{dRe}\,(\alpha)\wedge\mathrm{dIm}\,(\alpha)\equiv\mathrm{d}^2\alpha$):

$$\langle m|\varrho|n\rangle = \frac{\pi}{\sqrt{m!n!}}\left(\sqrt{\frac{1+r}{1-r}}\right)^{m+n}\int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\,F_{(0,0,r)}(\alpha,\alpha^*)$$
$$\times\frac{2}{\pi(1-r)}\exp\left(-\frac{2\alpha\alpha^*}{1-r}\right)L_{m,n}\left(\frac{2\alpha}{\sqrt{1-r^2}},\frac{2\alpha^*}{\sqrt{1-r^2}}\right). \tag{5.2}$$

Note that the indices of the Laguerre 2D polynomials in the relations (5.1) and (5.2) are reversed in comparison to each other. To see the correctness of (5.1) for the quasiprobabilities in Fock-state representation, one can use their derivation and explicit representation given, e.g., in [1] in connection with the definition of the Laguerre 2D polynomials in the present paper or of Laguerre 2D functions in [2]. It is clear that many partial results are much older (see, e.g., [9,10]). In particular, the Wigner quasiprobability $W(\alpha,\alpha^*)$ corresponding to $r=0$ in (5.1) was already derived for Fock states in [6] and the coherent-state (or Husimi–Kano) quasiprobability $Q(\alpha,\alpha^*)$ corresponding to $r=1$ in (4.1) is very easy to obtain for all operators $|n\rangle\langle m|$ from the most simple definition of this quasiprobability by $Q(\alpha,\alpha^*)\equiv\langle\alpha|\varrho|\alpha\rangle/\pi$ given by Kano [8] in connection with the Fock-state representation of coherent states $|\alpha\rangle$. The Glauber–Sudarshan quasiprobability $P(\alpha,\alpha^*)$ [6,7] corresponding to $r=-1$ in (4.1) which is the most singular of the usual quasiprobabilities was given in Fock-state representation in [7] but the connection of this representation with the limiting case $r=-1$ in (5.1) is not very easy to establish [1].

The well known Fock-state representation of the Husimi–Kano quasiprobability

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\exp(-\alpha\alpha^*)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\frac{\alpha^n\alpha^{*m}}{\sqrt{m!n!}} \tag{5.3}$$

can be obtained from (5.1) for $r\to1$ by using $L_{n,m}(z,z^*)\approx z^n z^{*m}$ for $|z|\to\infty$ (see (2.1)). Its inversion can be directly obtained from the explicit form of the coefficients in this Taylor series expansion

$$\langle m|\varrho|n\rangle = \frac{\pi}{\sqrt{m!n!}}\left\{\frac{\partial^{m+n}}{\partial\alpha^{*m}\partial\alpha^n}\exp(\alpha\alpha^*)Q(\alpha,\alpha^*)\right\}_{\alpha=\alpha^*=0}$$
$$= \frac{\pi(-\mathrm{i})^{m+n}}{\sqrt{m!n!}}\left\{L_{m,n}\left(\mathrm{i}\frac{\partial}{\partial\alpha},\mathrm{i}\frac{\partial}{\partial\alpha^*}\right)Q(\alpha,\alpha^*)\right\}_{\alpha=\alpha^*=0}. \tag{5.4}$$

The form in the first line sometimes has an advantage in comparison with the form in the second line with disentangled derivatives of $Q(\alpha,\alpha^*)$, for example, for pure states $\varrho=|\psi\rangle\langle\psi|$ because then $\exp(\alpha\alpha^*)\pi Q(\alpha,\alpha^*)$ splits into a product of two analytic functions $\psi(\alpha^*)$ and $\psi^*(\alpha)$ which are the Bargmann representations of the state obtained by scalar multiplication of $|\psi\rangle$ and $\langle\psi|$ with the analytic (non-normalized) coherent states.

The Wigner quasiprobability $W(\alpha,\alpha^*)$ corresponding to $r=0$ is given in Fock-state representation by

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\exp(-2\alpha\alpha^*)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle\frac{1}{\sqrt{m!n!}}L_{n,m}(2\alpha,2\alpha^*) \tag{5.5}$$

with the inversion

$$\langle m|\varrho|n\rangle = \frac{2}{\sqrt{m!n!}}\int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\,W(\alpha,\alpha^*)\exp(-2\alpha\alpha^*)L_{m,n}(2\alpha,2\alpha^*). \tag{5.6}$$

This shows that the Wigner quasiprobability $W(\alpha, \alpha^*)$ is most simply connected with the Laguerre 2D polynomials and 2D functions.

The Glauber–Sudarshan quasiprobability $Q(\alpha, \alpha^*)$ corresponding to $r = -1$ has to be extracted from (5.1) by the limiting procedure $\varepsilon \to 0$ after setting $r = -\sqrt{1-4\varepsilon}$ and by using (4.3) and (4.6) with the result [1] ([19, 20] for Fock states):

$$
\begin{aligned}
P(\alpha, \alpha^*) &= \exp(\alpha\alpha^*) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{(-1)^{m+n}}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial\alpha^m \partial\alpha^{*n}} \delta(\alpha, \alpha^*) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \langle m|\varrho|n\rangle \frac{\mathrm{i}^{m+n}}{\sqrt{m!n!}} L_{m,n}\left(\mathrm{i}\frac{\partial}{\partial\alpha}, \mathrm{i}\frac{\partial}{\partial\alpha^*}\right) \delta(\alpha, \alpha^*).
\end{aligned}
\tag{5.7}
$$

The inversion of this relation provides

$$
\langle m|\varrho|n\rangle = \frac{1}{\sqrt{m!n!}} \int \frac{\mathrm{i}}{2} \, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* \, P(\alpha, \alpha^*) \exp(-\alpha\alpha^*) \alpha^m \alpha^{*n}. \tag{5.8}
$$

Instead of the 2D delta function $\delta(\alpha, \alpha^*)$, one can use in (5.7) a central-symmetric 1D delta function that was originally made by Sudarshan [7]. The connection to the form (5.7) was discussed in detail in [1] and the disentangled form for the multiplication of the delta function with a Gaussian function, both in the 2D form and in the central-symmetric 1D form, was given.

6. Quasiprobabilities in representation by ordered moments and their inversion

The reconstruction of the density operator ϱ (or of an arbitrary other operator) from its normally ordered moments $\langle \varrho a^{\dagger k} a^l \rangle$ was derived in [21] (see also [45–47]) and possesses the following form ($\langle A \rangle \equiv \mathrm{Trace}(A)$)

$$
\varrho = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{k,l} \langle \varrho a^{\dagger k} a^l \rangle \qquad a_{k,l} \equiv \sum_{j=0}^{\{k,l\}} \frac{(-1)^j |l-j\rangle\langle k-j|}{j!\sqrt{(k-j)!(l-j)!}}. \tag{6.1}
$$

There is a duality of this relation to the expansion of an operator ϱ in normal ordering of powers of boson annihilation and creation operator $(a, a^\dagger)$ as follows:

$$
\varrho = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a_{k,l} \rangle a^{\dagger k} a^l \tag{6.2}
$$

with the same set of auxiliary operators $a_{k,l}$ as in (6.1). Relation (6.1) shows that the complete set of normally ordered moments $\langle \varrho a^{\dagger k} a^l \rangle$ (or any otherwise ordered moments) contains the complete information of the density operator ϱ. This can be formulated in 'pure' form as a completeness relation for the dual sets of basis operators $a^{\dagger k} a^l$ and of $a_{k,l}$ which is similar to dual sets of contravariant and covariant basis vectors in a linear space [21]. Therefore, any other complete (or overcomplete) set of information about the density operator as, for example, the quasiprobabilities can be expressed by these moments.

From (6.1), one obtains the diagonal coherent-state representation which in the case of the density operator ϱ is proportional to the coherent-state quasiprobability $Q(\alpha, \alpha^*) \equiv \langle \alpha|\varrho|\alpha\rangle/\pi$ and can be expressed by the usual Laguerre polynomials ([21] equations (3.6)–(3.8) or (3.12)) but can be represented with advantage by the Laguerre 2D polynomials or the Laguerre 2D functions as follows:

$$
\begin{aligned}
Q(\alpha, \alpha^*) &= \frac{1}{\pi} \exp(-\alpha\alpha^*) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a^{\dagger k} a^l \rangle \frac{1}{k!l!} L_{k,l}(\alpha, \alpha^*) \\
&= \frac{1}{\sqrt{\pi}} \exp\left(-\frac{\alpha\alpha^*}{2}\right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a^{\dagger k} a^l \rangle \frac{1}{\sqrt{k!l!}} l_{k,l}(\alpha, \alpha^*).
\end{aligned}
\tag{6.3}
$$

The last relation can be inverted by using the orthonormality of the Laguerre 2D functions providing the normally ordered moments in dependence on $Q(\alpha,\alpha^*)$ ([21] equation (3.10) in a more general operator form; see also [9, 25, 48])

$$\begin{aligned}\langle\varrho a^{\dagger k}a^l\rangle &= \sqrt{\pi k!l!}\int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\,Q(\alpha,\alpha^*)\exp\left(\frac{\alpha\alpha^*}{2}\right)l_{l,k}(\alpha,\alpha^*)\\ &= \int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\,Q(\alpha,\alpha^*)L_{l,k}(\alpha,\alpha^*)\qquad Q(\alpha,\alpha^*)\equiv\left\langle\varrho\frac{|\alpha\rangle\langle\alpha|}{\pi}\right\rangle.\end{aligned}\tag{6.4}$$

We see that the most simple expression (without a Gaussian factor) is obtained by using the Laguerre 2D polynomials instead of the Laguerre 2D functions. The last two and similar relations can be represented as pure operator identities by separation of the arbitrary density operator ϱ [21]. The operators $T_r(\alpha,\alpha^*)$ obtained by separation of the density operator ϱ from quasiprobabilities $F_r(\alpha,\alpha^*)=\langle\varrho\,T_r(\alpha,\alpha^*)\rangle$ are called transition operators [36].

The generalization of (6.3) to the whole class of quasiprobabilities $F_{(0,0,r)}(\alpha,\alpha^*)$ can be made, for example, by using the relation (see, e.g., [36], equations (5.1), (5.4) and (5.7))

$$F_{(0,0,r)}(\alpha,\alpha^*)=\exp\left(-\frac{1-r}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)Q(\alpha,\alpha^*).\tag{6.5}$$

By applying the alternative representation (2.4) of the Laguerre 2D polynomials and after changing the order of the commutative products of two operators with partial derivatives, one obtains the intermediate result

$$\begin{aligned}F_{(0,0,r)}(\alpha,\alpha^*) &= \frac{1}{\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle\varrho a^{\dagger k}a^l\rangle\frac{(-1)^{k+l}}{k!l!}\frac{\partial^{k+l}}{\partial\alpha^{*k}\partial\alpha^l}\exp\left(-\frac{1-r}{2}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right)\exp(-\alpha\alpha^*)\\ &= \frac{2}{\pi(1+r)}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle\varrho a^{\dagger k}a^l\rangle\frac{(-1)^{k+l}}{k!l!}\frac{\partial^{k+l}}{\partial\alpha^{*k}\partial\alpha^l}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\end{aligned}\tag{6.6}$$

which, by again using (2.4), can be represented by means of the Laguerre 2D polynomials in the final form

$$\begin{aligned}F_{(0,0,r)}(\alpha,\alpha^*) &= \frac{2}{\pi(1+r)}\exp\left(-\frac{2\alpha\alpha^*}{1+r}\right)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle\varrho a^{\dagger k}a^l\rangle\\ &\quad\times\frac{1}{k!l!}\left(\sqrt{\frac{2}{1+r}}\right)^{k+l}L_{k,l}\left(\sqrt{\frac{2}{1+r}}\alpha,\sqrt{\frac{2}{1+r}}\alpha^*\right)\end{aligned}\tag{6.7}$$

or by using the Laguerre 2D functions

$$\begin{aligned}F_{(0,0,r)}(\alpha,\alpha^*) &= \frac{2}{\sqrt{\pi}(1+r)}\exp\left(-\frac{\alpha\alpha^*}{1+r}\right)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\langle\varrho a^{\dagger k}a^l\rangle\\ &\quad\times\frac{1}{\sqrt{k!l!}}\left(\sqrt{\frac{2}{1+r}}\right)^{k+l}l_{k,l}\left(\sqrt{\frac{2}{1+r}}\alpha,\sqrt{\frac{2}{1+r}}\alpha^*\right).\end{aligned}\tag{6.8}$$

The inversion can be obtained by using the orthonormality of the Laguerre 2D functions with the result

$$\begin{aligned}\langle\varrho a^{\dagger k}a^l\rangle &= \sqrt{\pi k!l!}\left(\sqrt{\frac{1+r}{2}}\right)^{k+l}\int\frac{\mathrm{i}}{2}\,\mathrm{d}\alpha\wedge\mathrm{d}\alpha^*\,F_{(0,0,r)}(\alpha,\alpha^*)\\ &\quad\times\exp\left(\frac{\alpha\alpha^*}{1+r}\right)l_{l,k}\left(\sqrt{\frac{2}{1+r}}\alpha,\sqrt{\frac{2}{1+r}}\alpha^*\right)\end{aligned}\tag{6.9}$$

or written by the Laguerre 2D polynomials (cf [48], equations (8) and (9))

$$\langle \varrho a^{\dagger k} a^l \rangle = \left(\sqrt{\frac{1+r}{2}} \right)^{k+l} \int \frac{\mathrm{i}}{2}\, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* F_{(0,0,r)}(\alpha, \alpha^*) L_{l,k} \left(\sqrt{\frac{2}{1+r}}\alpha, \sqrt{\frac{2}{1+r}}\alpha^* \right). \tag{6.10}$$

We do not have a Gaussian factor under the integral and the normally ordered moments are almost directly obtained from the integral over the quasiprobabilities multiplied with Laguerre 2D polynomials. In particular, in the case of the Wigner quasiprobability ($r = 0$) one has

$$W(\alpha, \alpha^*) = \frac{2}{\pi} \exp(-2\alpha\alpha^*) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a^{\dagger k} a^l \rangle \frac{(\sqrt{2})^{k+l}}{k!l!} L_{k,l}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) \tag{6.11}$$

with the inversion

$$\langle \varrho a^{\dagger k} a^l \rangle = \frac{1}{(\sqrt{2})^{k+l}} \int \frac{\mathrm{i}}{2}\, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* W(\alpha, \alpha^*) L_{l,k}(\sqrt{2}\alpha, \sqrt{2}\alpha^*). \tag{6.12}$$

The Glauber–Sudarshan quasiprobability $P(\alpha, \alpha^*)$ corresponding to $r = -1$ has to be obtained from (6.8) by a limiting procedure $\varepsilon \to 0$ after setting $r = -1 + 2\varepsilon$ and by using (4.3) with the result

$$P(\alpha, \alpha^*) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho a^{\dagger k} a^l \rangle \frac{(-1)^{k+l}}{k!l!} \frac{\partial^{k+l}}{\partial \alpha^{*k} \partial \alpha^l} \delta(\alpha, \alpha^*). \tag{6.13}$$

The inversion of (6.13) following from (6.10) for $r \to -1$ is the well known relation

$$\langle \varrho a^{\dagger k} a^l \rangle = \int \frac{\mathrm{i}}{2}\, \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* P(\alpha, \alpha^*) \alpha^{*k} \alpha^l. \tag{6.14}$$

Both relations (6.13) and (6.14) can be more directly obtained from the following definition of $P(\alpha, \alpha^*)$ [36]:

$$P(\alpha, \alpha^*) = \left\langle \varrho \exp\left(-a^\dagger \frac{\partial}{\partial \alpha^*} \right) \exp\left(-a \frac{\partial}{\partial \alpha} \right) \right\rangle \delta(\alpha, \alpha^*) \tag{6.15}$$

and from its inversion

$$\varrho = \int \frac{\mathrm{i}}{2} \mathrm{d}\alpha \wedge \mathrm{d}\alpha^* P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| \tag{6.16}$$

which mostly, however, serves as the primary definition of $P(\alpha, \alpha^*)$ [6, 7]. By Taylor series expansion of the exponential operators in the first and by forming $\langle \varrho a^{\dagger k} a^l \rangle$ in the second equation, one obtains (6.13) and (6.14).

The transition between different kinds of ordering itself within the line of normal ordering $r = -1$ through symmetrical ordering $r = 0$ to antinormal ordering $r = 1$ can also be expressed with advantage by means of the Laguerre 2D polynomials ($\mathcal{O}_{\boldsymbol{r}}\{\ldots\}$ ordering symbol in analogy to $F_{\boldsymbol{r}}(\alpha, \alpha^*)$)

$$\begin{aligned} \mathcal{O}_{(0,0,s)}\{a^{\dagger k} a^l\} &= \sum_{j=0}^{\{k,l\}} \frac{k!l!}{j!(k-j)!(l-j)!} \left(-\frac{r-s}{2} \right)^j \mathcal{O}_{(0,0,r)}\{a^{\dagger k-j} a^{l-j}\} \\ &= \left(\sqrt{\frac{r-s}{2}} \right)^{k+l} \mathcal{O}_{(0,0,r)} \left\{ L_{k,l} \left(\sqrt{\frac{2}{r-s}} a^\dagger, \sqrt{\frac{2}{r-s}} a \right) \right\}. \end{aligned} \tag{6.17}$$

By using the transformation relations for the Laguerre 2D polynomials in (3.19), one can generalize (6.7) to

$$\begin{aligned} F_{(0,0,r)}(\alpha, \alpha^*) &= \frac{2}{\pi(r-s)} \exp\left(-\frac{2\alpha\alpha^*}{r-s} \right) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle \varrho \mathcal{O}_{(0,0,s)}\{a^{\dagger k} a^l\} \rangle \\ &\quad \times \frac{1}{k!l!} \left(\sqrt{\frac{2}{r-s}} \right)^{k+l} L_{k,l} \left(\sqrt{\frac{2}{r-s}} \alpha, \sqrt{\frac{2}{r-s}} \alpha^* \right) \end{aligned} \tag{6.18}$$

with the inversion

$$\langle \varrho \mathcal{O}_{(0,0,s)}\{a^{\dagger k} a^l\}\rangle = \left(\sqrt{\frac{r-s}{2}}\right)^{k+l} \int \frac{\mathrm{i}}{2}\,\mathrm{d}\alpha \wedge \mathrm{d}\alpha^*\, F_{(0,0,r)}(\alpha,\alpha^*) \times L_{l,k}\left(\sqrt{\frac{2}{r-s}}\alpha, \sqrt{\frac{2}{r-s}}\alpha^*\right). \tag{6.19}$$

Thus we expressed the quasiprobabilities of the class $F_{(0,0,r)}(\alpha,\alpha^*)$ by ordered moments of the class $\langle \varrho\, \mathcal{O}_{(0,0,s)}\{a^{\dagger k} a^l\}\rangle$ and have obtained the inversion. In cases $r = s$, one has to make limiting procedures with results in analogy to (6.13) and (6.14) with the substitutions $P(\alpha,\alpha^*) \to F_{(0,0,r)}(\alpha,\alpha^*)$ and $\langle \varrho\, a^{\dagger k} a^l\rangle \to \langle \varrho\, \mathcal{O}_{(0,0,r)}\{a^{\dagger k} a^l\}\rangle$, for example $W(\alpha,\alpha^*)$ and $\langle \varrho\, \mathcal{S}\{a^{\dagger k} a^l\}\rangle$ in the case of symmetrical ordering.

The results of this section show the appropriateness of the Laguerre 2D polynomials for problems of quantum optics connected with operator ordering.

7. Transformations of the quasiprobabilities

In this section we apply transformation formulae for the Laguerre 2D polynomials and their limiting cases derived in sections 3 and 4. There are many possibilities and we show only some principal cases without specialization in all details.

We begin with the Peřina–Mišta representation [12, 13, 22–24] which is a regularized representation of the Glauber–Sudarshan quasiprobability $P(\alpha,\alpha^*)$ reformulated in [1, 2]. By inserting the 'regularized' representation of the derivatives of the 2D delta function by means of the Laguerre 2D polynomials in (4.3) into formula (5.7) for $P(\alpha,\alpha^*)$, after substitutions of the summation indices, one finds

$$P(\alpha,\alpha^*) = \exp(\alpha\alpha^*)\frac{1}{\pi\varepsilon}\exp\left(-\frac{\alpha\alpha^*}{\varepsilon}\right)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varrho_{m,n}(\varepsilon)}{\sqrt{m!n!}(\sqrt{\varepsilon})^{m+n}}L_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right) \tag{7.1}$$

with the following abbreviation for effective matrix elements $\varrho_{m,n}(\varepsilon)$ and their inversion

$$\begin{aligned}
\varrho_{m,n}(\varepsilon) &\equiv \sum_{j=0}^{\{m,n\}}\frac{(-\varepsilon)^j\sqrt{m!n!}}{j!\sqrt{(m-j)!(n-j)!}}\,\varrho_{m-j,n-j}(0) = \langle m|\left(\sum_{j=0}^{\infty}\frac{(-\varepsilon)^j}{j!}a^{\dagger j}\varrho a^j\right)|n\rangle \\
\varrho_{m,n}(0) &= \sum_{j=0}^{\{m,n\}}\frac{\varepsilon^j\sqrt{m!n!}}{j!\sqrt{(m-j)!(n-j)!}}\,\varrho_{m-j,n-j}(\varepsilon) \equiv \langle m|\varrho|n\rangle.
\end{aligned} \tag{7.2}$$

These formulae for the effective matrix elements possess the form of the more general transformations (3.14) and (3.15) with $f_{m,n}(0; z, z^*) = \varrho_{m,n}(0)/\sqrt{m!n!}$ and $\mu = 1$ and the inversion becomes immediately obvious. In comparison to our former derivation [1, 2] or to the derivation in [22, 23], we have extracted and expressed the essence of such derivations in the pure mathematical form as the transformation formulae (4.2) or (4.4) for the representation of Laguerre 2D polynomials by derivatives of the 2D delta function and their inversion. Since it is easy to find the inversion of (7.1) that means the effective matrix elements $\varrho_{m,n}(\varepsilon)$ in dependence on $P(\alpha,\alpha^*)$, we do not write it down (see section 5 for analogous inversions). The new effective matrix elements $\varrho_{m,n}(\varepsilon)$ are not normalized in the same way as $\varrho_{m,n}(0)$ as the following short calculation clarifies:

$$\sum_{n=0}^{\infty}\varrho_{n,n}(\varepsilon) = \frac{1}{1+\varepsilon}\sum_{m=0}^{\infty}\frac{\varrho_{m,m}(0)}{(1+\varepsilon)^m} \qquad \sum_{n=0}^{\infty}\varrho_{n,n}(0) = 1. \tag{7.3}$$

This normalization depends not only on ε but also on the matrix elements $\varrho_{m,m}(0)$. A (normalized) effective density operator $\varrho(\varepsilon)$ which could be defined by the content in brackets between the Fock states $\langle m|$ and $|n\rangle$ in (7.2) is not always positively definite.

As another example, we consider the transformation of the Husimi–Kano quasiprobability $Q(\alpha,\alpha^*)$ in Fock-state representation given in (5.3) by means of (3.21). This provides in a first step

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\exp(-\alpha\alpha^*)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle m|\varrho|n\rangle \times \frac{(\sqrt{\varepsilon})^{m+n}}{\sqrt{m!n!}}\sum_{j=0}^{\{m,n\}}\frac{m!n!}{j!(m-j)!(n-j)!}L_{n-j,m-j}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right). \tag{7.4}$$

One can also look to this expression of $Q(\alpha,\alpha^*)$ by Laguerre 2D polynomials in the following way. The functions connected with the matrix elements $\langle m|\varrho|n\rangle$ can be expressed by one of the quasiprobabilities of the class $F_{(0,0,r)}(\alpha,\alpha^*)$ with stretched arguments taken for certain superpositions of lower matrix elements $\langle m-j|\varrho|n-j\rangle$, $(j=0,1,\ldots,\{m,n\})$. If one chooses $F_{(0,0,1-2\varepsilon)}(\sqrt{1-\varepsilon}\alpha,\sqrt{1-\varepsilon}\alpha^*)$ for given $\varepsilon\leqslant 1$, one can express $Q(\alpha,\alpha^*)$ for any matrix element $\langle m|\varrho|n\rangle$ by the quasiprobabilities $F_{(0,0,1-2\varepsilon)}(\sqrt{1-\varepsilon}\alpha,\sqrt{1-\varepsilon}\,\alpha^*)$ for a superposition of matrix elements $\langle m-j|\varrho|n-j\rangle$ with full absorption of the Gaussian factor $\exp(-\alpha\alpha^*)$ in $F_{(0,0,1-2\varepsilon)}(\sqrt{1-\varepsilon}\alpha,\sqrt{1-\varepsilon}\alpha^*)$. By choosing $\varepsilon=\frac{1}{2}$, one obtains an equivalence to the Wigner quasiprobability $W(\alpha/\sqrt{2},\alpha^*/\sqrt{2})$ for a certain substitute of matrix elements. This was made in [25] for the diagonal elements $\langle n|\varrho|n\rangle$ of the density operator together with the inversion. The generalization for nondiagonal elements is contained in (7.4) by choosing $\varepsilon=\frac{1}{2}$ and by using (5.5). The inversion is contained as a special case in the generalization considered below but it is easy to obtain for diagonal elements.

We consider another aspect of (7.4). By introduction of new effective matrix elements in a second step after substitutions of the summation indices, one can write (7.4) in the form

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\exp(-\alpha\alpha^*)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varrho_{m,n}(\varepsilon)}{\sqrt{m!n!}}(\sqrt{\varepsilon})^{m+n}L_{n,m}\left(\frac{\alpha}{\sqrt{\varepsilon}},\frac{\alpha^*}{\sqrt{\varepsilon}}\right) \tag{7.5}$$

with the following definition of $\varrho_{m,n}(\varepsilon)$ together with its inversion (different from (7.2))

$$\begin{aligned}\varrho_{m,n}(\varepsilon) &\equiv \sum_{k=0}^{\infty}\frac{\varepsilon^k\sqrt{(m+k)!(n+k)!}}{k!\sqrt{m!n!}}\varrho_{m+k,n+k}(0) = \langle m|\left(\sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}a^k\varrho a^{\dagger k}\right)|n\rangle\\ \varrho_{m,n}(0) &= \sum_{k=0}^{\infty}\frac{(-\varepsilon)^k\sqrt{(m+k)!(n+k)!}}{k!\sqrt{m!n!}}\varrho_{m+k,n+k}(\varepsilon) \equiv \langle m|\varrho|n\rangle.\end{aligned} \tag{7.6}$$

In contrast to (7.2), the new effective matrix elements $\varrho_{m,n}(\varepsilon)$ contain contributions from all Fock-state matrix elements above $\langle m|\varrho|n\rangle$ for given (m,n). The normalization of the new effective matrix elements is clarified by the relation

$$\sum_{n=0}^{\infty}\varrho_{n,n}(\varepsilon) = \sum_{m=0}^{\infty}(1+\varepsilon)^m\varrho_{m,m}(0). \tag{7.7}$$

If one forms an effective (nonnormalized) ‘density’ operator $\varrho(\varepsilon)$ according to

$$\varrho(\varepsilon) = \sum_{k=0}^{\infty}\frac{\varepsilon^k}{k!}a^k\varrho a^{\dagger k} \tag{7.8}$$

then it is not always normalizable for positive ε. For example, for thermal states ϱ_{th} with mean value $\bar{N}$ of the number operator $N = a^\dagger a$, one has

$$\begin{aligned} \varrho_{\mathrm{th}} &= \frac{1}{1+\bar{N}} \sum_{n=0}^{\infty} \left(\frac{\bar{N}}{1+\bar{N}}\right)^n |n\rangle\langle n| \qquad \sum_{n=0}^{\infty} \varrho_{n,n}(0) = 1 \\ \varrho(\varepsilon) &= \frac{1}{1+\bar{N}-\varepsilon\bar{N}} \sum_{n=0}^{\infty} \left(\frac{\bar{N}}{1+\bar{N}-\varepsilon\bar{N}}\right)^n |n\rangle\langle n| \qquad \sum_{n=0}^{\infty} \varrho_{n,n}(\varepsilon) = \frac{1}{1-\varepsilon\bar{N}} \end{aligned} \tag{7.9}$$

which is convergent for positive ε only under the condition $\varepsilon\bar{N} < 1$ and is therefore only in this case a trace-class operator. Nevertheless, it seems that (7.5) together with (7.6) can be used for a 'restricted' design of quasiprobabilities, for example, if one looks for states which provide a certain desired quasiprobability $Q(\alpha, \alpha^*)$ in the form of a simple superposition of Laguerre 2D polynomials. By using the orthonormality relations for the Laguerre 2D functions, one first determines the effective matrix elements $\varrho_{m,n}(\varepsilon)$ and then by the inverse relations the necessary Fock-state matrix elements $\langle m|\varrho|n\rangle$. We call this 'restricted' design for the following reason. In principle, for any given quasiprobability $F_{(0,0,r)}(\alpha, \alpha^*)$, in particular $W(\alpha, \alpha^*)$ or $Q(\alpha, \alpha^*)$, one can determine the necessary Fock-state matrix elements $\langle m|\varrho|n\rangle$ by the relation (5.2) but this way can be troublesome if the functions are weakly related to the Laguerre 2D polynomials. It is necessary to underline that, if one has found an effective density operator $\varrho(\varepsilon)$ not contradicting the conditions for density operators, then (7.8) does not provide a recipe for its realization because the transformation $\varrho \to \varrho_{\mathrm{norm}}(\varepsilon)$ is not a unitary transformation but is a superoperator acting on ϱ.

By using the more general transformation of the Laguerre 2D polynomials in (3.24), one can transform the class of quasiprobabilities $F_{(0,0,r)}(\alpha, \alpha^*)$ (5.1) to the representation

$$\begin{aligned} F_{(0,0,r)}(\alpha, \alpha^*) = {}& \frac{2}{\pi(1+r)} \exp\left(-\frac{2\alpha\alpha^*}{1+r}\right) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varrho_{m,n}(\varepsilon) \frac{1}{\sqrt{m!n!}} \left(\sqrt{\frac{1-r}{1+r}+\varepsilon}\right)^{m+n} \\ & \times L_{n,m}\left(\frac{2\alpha}{\sqrt{(1+r)(1-r+\varepsilon(1+r))}}, \frac{2\alpha^*}{\sqrt{(1+r)(1-r+\varepsilon(1+r))}}\right) \end{aligned} \tag{7.10}$$

with the same effective matrix elements as in (7.6), independently on r

$$\begin{aligned} \varrho_{m,n}(\varepsilon) &\equiv \sum_{k=0}^{\infty} \frac{\varepsilon^k \sqrt{(m+k)!(n+k)!}}{j!\sqrt{m!n!}} \langle m+k|\varrho|n+k\rangle \\ \langle m|\varrho|n\rangle &\equiv \sum_{k=0}^{\infty} \frac{(-\varepsilon)^k \sqrt{(m+k)!(n+k)!}}{j!\sqrt{m!n!}} \varrho_{m+k,n+k}(\varepsilon) \end{aligned} \tag{7.11}$$

which is identical with (7.5) and (7.6) for $r = 1$. The most common representation (5.1) of the quasiprobabilities is obtained for $\varepsilon = 0$. In particular, alternative representations of the Wigner quasiprobability $W(\alpha, \alpha^*)$ which are contained in (7.10) as the special case $r = 0$ seem to be interesting. As was already said for the case of the coherent-state quasiprobaility $Q(\alpha, \alpha^*)$, a 'restricted' design of Wigner quasiprobabilities given by simple combinations of Laguerre 2D functions with stretched arguments or by products of Hermite functions and the determination of the corresponding density operators should become possible.

8. Conclusion

We have introduced Laguerre 2D polynomials and have considered some of their most essential properties. In particular, we have derived transformation formulae for the Laguerre 2D polynomials (and also Hermite polynomials) from a more general kind of transformation applicable to some classes of polynomials. Furthermore, we derived limiting cases of the Laguerre 2D polynomials and related regularized representations of the derivatives of the 2D delta function. The transformation formulae between Laguerre 2D polynomials also seem to be useful for extending the possibilities of integration.

We applied the Laguerre 2D polynomials and 2D functions to the solution of some problems of quantum optics. In particular, we have represented the quasiprobabilities in Fock-state basis and in representation by the normally ordered moments and by more generally ordered moments whereby the inversion of these formulae is obtained. The application of the derived transformation formulae leads to representations of the quasiprobabilities by Laguerre 2D polynomials with stretched arguments and with the stretch factor as a free parameter and to deformed effective Fock-state matrix elements. All these considerations show the appropriateness of the introduced Laguerre 2D polynomials and Laguerre 2D functions.

The Laguerre 2D functions are eigenstates of the degenerate 2D harmonic oscillator. The Hamilton operator of this system is invariant with regard to $SU(2)$ transformations of an orthonormal basis if one splits the global $U(1)$ invariance of quantum mechanics from the more general $U(2) = U(1) \times SU(2)$ invariance of the harmonic 2D oscillator. The consideration of the transformations of $SU(2)$ or, more generally, of $SL(2, C)$ leads to the introduction of a more general (real or complex) three-parameter class of 2D polynomials and 2D functions which includes the Laguerre 2D polynomials considered here as well as products of two Hermite polynomials as special cases and possesses a close connection to the two-variable Hermite polynomials. The essential point is the generalization of the generating function (2.10) or the generalization of equation (3.4) in [2] in application to powers of more general linear combinations of (z, z^*) or (x, y), where Jacobi polynomials $P_j^{(\alpha,\beta)}(u)$ appear in the transition formulae not only with the argument $u = 0$ but with real arguments between -1 and +1 (case $SU(2)$) or with complex arguments. This appears to complete the introduction of different sets of orthonormalized functions connected with the degenerate 2D harmonic oscillator and leads to their unification. The embedding of the dynamical group $SU(2)$ of a 2D harmonic oscillator into the general ten-parameter symplectic group $Sp(4, R)$ of a two-mode system which is equivalent to the De-Sitter group $SO(3, 2)$ [18] with different subgroups $SU(1, 1)$ of squeezing and the proper Lorentz group as another subgroup should reveal relations to many other problems of quantum theory.

The Laguerre 2D polynomials and 2D functions can also be applied with advantage in classical optics for the treatment of Gauss–Laguerre beams which are solutions of the wave equation in the paraxial approximation. In the case of Gauss–Laguerre beams, one has as a third variable, the propagation direction, which is usually denoted by z whereas $z \equiv x + iy$, $z^* \equiv x - iy$ in our treatment of Laguerre 2D functions is related to the (xy)-plane. One can overcome this conflict by renaming the variables (z, z^*) in our treatment of the Laguerre 2D polynomials by $(z, z^*) \to (z_+, z_-)$. I had this in mind together with the treatment of right-hand and left-hand circular polarization, when I denoted the annihilation and creation operators introduced in [2] by (a_+, a_-) and $(a_+^\dagger, a_-^\dagger)$.

The obtained simple relations in analogy to relations for Hermite polynomials together with the fundamental applications in quantum optics justify the introduction of the Laguerre 2D polynomials beside the Laguerre 2D functions introduced and discussed in [1, 2].

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Appendix A. A supplement to associated Laguerre polynomials

In this appendix I make a supplement about one relation of the associated (or generalized) Laguerre polynomials $L_n^\alpha(u)$ to the special Laguerre polynomials $L_n(u) \equiv L_n^0(u)$ which seems to be absent in the most popular and most often used collections of formulae and tables. The associated Laguerre polynomials are defined by [26, 33–35]

$$L_n^\alpha(u) \equiv \frac{1}{n!} u^{-\alpha} \mathrm{e}^u \frac{\partial^n}{\partial u^n} \mathrm{e}^{-u} u^{n+\alpha} = \frac{1}{n!} u^{-\alpha} \left(\frac{\partial}{\partial u} - 1 \right)^n u^{n+\alpha} \tag{A.1}$$

and are explicitly given by

$$L_n^\alpha(u) = \sum_{j=0}^{n} \frac{(n+\alpha)!}{j!(n+\alpha-j)!(n-j)!} (-u)^{n-j} = \sum_{k=0}^{n} \frac{(\alpha+n)!}{k!(\alpha+k)!(n-k)!} (-u)^k. \tag{A.2}$$

For integer α or by using fractal differentiation or integration for arbitrary α, (A.1) can be substituted by

$$L_n^\alpha(u) = \frac{(-1)^n}{n!} \mathrm{e}^u \left(-\frac{\partial}{\partial u} \right)^{n+\alpha} \mathrm{e}^{-u} u^n = \frac{(-1)^n}{n!} \left(1 - \frac{\partial}{\partial u} \right)^{n+\alpha} u^n \tag{A.3}$$

with

$$\left(1 - \frac{\partial}{\partial u} \right)^{n+\alpha} \equiv \sum_{j=0}^{\infty} \frac{(-1)^j (n+\alpha)!}{j!(n+\alpha-j)!} \frac{\partial^j}{\partial u^j} \tag{A.4}$$

in the sense of the Taylor series expansion of the function $(1-x)^{n+\alpha}$. One can prove (A.3) by explicit calculation of the derivatives of u^n according to the expansion (A.4) and by comparison of the result with (A.2). I remark in this connection that (A.1) and (A.3) considered as differential or integral operators are not equivalent and lead to the same result only in application to functions proportional to $f(u) = 1$ as one easily finds by choosing $f(u) = u^m$. Therefore, an identity of (A.1) and (A.3) cannot be proved on the operator level. The representation (A.3) leads to the following new connection of the associated Laguerre polynomials $L_n^\alpha(u)$ with the special case $L_n(u) \equiv L_n^0(u)$:

$$L_n^\alpha(u) = \left(1 - \frac{\partial}{\partial u} \right)^\alpha L_n(u). \tag{A.5}$$

On the other hand, one easily obtains for integer α the following known representation of $L_n^\alpha(u)$ by $L_{n+\alpha}(u)$:

$$L_n^\alpha(u) = \left(-\frac{\partial}{\partial u} \right)^\alpha L_{n+\alpha}(u). \tag{A.6}$$

Relation (A.5) possesses the advantage in comparison to (A.6) that it also works in the case of fractal α.

I mention here that some older textbooks and tables, for example, Morse and Feshbach [49] and, up to the third Russian edition Gradshteyn and Ryzhik [34], use slightly different definitions of the associated Laguerre polynomials (moreover, the definitions in [49] and in older editions of [34] are also different from each other; from the fourth Russian edition on, [34] uses the modern definition given in (A.1) and this should be similar in the many later translations of [34]). The formulae of quantum optics with Laguerre polynomials in the monographs of Peřina [12,13] use the now rarely applied definition of Morse and Feshbach [49]. Furthermore, I mention that some now comparatively old monographs, e.g., [50–52] define the associated Laguerre polynomials $L_n^\alpha(u)$ or sometimes denoted by $L_n^{(\alpha)}(u)$ by α-fold differentiation of $L_n(u)$ whereas in the new definition they are obtained according to (A.6) by

α-fold differentiation of $L_{n+\alpha}(u)$ and multiplication with $(-1)^\alpha$ (for integer α) or according to (A.5) by application of the operator $(1 - \partial/\partial u)^\alpha$ to $L_n(u)$ that is possible not only for integer α but for fractal α too. All this means that in the modern definition, the maximal power of u in $L_n^\alpha(u)$ is u^n independently on α with the coefficient $(-1)^n/n!$ in front which is also independent on α. Additionally, I mention that the notion 'Laguerre polynomials' for $L_n(u)$ and 'associated Laguerre polynomials' for $L_n^\alpha(u)$ are not uniform in the literature and that the last are often called 'generalized Laguerre polynomials' corresponding to the historical fact that Laguerre did not investigate them.

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